ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $y dx - x dy + 3x^2 y^2 e^{x^3} dx = 0$.

Sol. Since $3x^2 e^{x^3} = d(e^{x^3})$, the term $3x^2 y^2 e^{x^3} dx$ should not involve y^2.

This suggests that $\frac{1}{y^2}$ may be an I.F.

Multiplying throughout by $\frac{1}{y^2}$, we have $\frac{y dx - x dy}{y^2} + 3x^2 e^{x^3} dx = 0$

or $$d\left(\frac{x}{y}\right) + d(e^{x^3}) = 0, \text{ which is exact.}$$

Integrating, we get $\frac{x}{y} + e^{x^3} = c$, which is the required solution.

Example 2. *Solve* $x dy - y dx = x\sqrt{x^2 - y^2}\, dx$.

Sol. The given equation is $x dy - y dx = x^2 \sqrt{1 - \left(\frac{y}{x}\right)^2}\, dx$ or $\frac{\frac{x dy - y dx}{x^2}}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} = dx$

or $$d\left(\sin^{-1}\frac{y}{x}\right) = dx, \text{ which is exact.}$$

Integrating, we get $\sin^{-1}\frac{y}{x} = x + c$ or $y = x \sin(x + c)$, which is the required solution.

Example 3. *Solve:* $x dx + y dy = \frac{a^2(x dy - y dx)}{x^2 + y^2}$.

Sol. The given equation is $x dx + y dy - a^2 d\left(\tan^{-1}\frac{y}{x}\right) = 0$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} - a^2 \tan^{-1}\frac{y}{x} = c$

or $$x^2 + y^2 - 2a^2 \tan^{-1}\frac{y}{x} = C, \quad \text{where } C = 2c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $x dy - y dx = (x^2 + y^2)\, dx$
2. $x dy - y dx = (x^2 + y^2)(dx + dy)$
3. $y(2xy + e^x)\, dx = e^x\, dy$
4. $(y \log y - 2xy)\, dx + (x + y)\, dy = 0$
5. $x dy - y dx = xy^2\, dx$
6. $x dy = (x^2y^2 - y)\, dx$
7. $(x + y)^2\left(x\frac{dy}{dx} + y\right) = xy\left(1 + \frac{dy}{dx}\right)$
8. $x dy - y dx = (4x^2 + y^2) dy$
9. $(y + y^2 \cos x)\, dx - (x - y^3)\, dy = 0$.

Answers

1. $\tan^{-1}\frac{y}{x} = x + c$

2. $\tan^{-1}\frac{y}{x} = x + y + c$

3. $x^2 + \frac{e^x}{y} = c$

4. $x \log y - x^2 + y = c$

5. $\frac{x}{y} + \frac{x^2}{2} = c$

6. $-\frac{1}{xy} = x + c$

7. $\log(xy) = -\frac{1}{x+y} + c$

8. $\frac{1}{2}\tan^{-1}\left(\frac{y}{2x}\right) = y + c$

9. $\frac{x}{y} + \sin x + \frac{y^2}{2} = c.$

Hints

1. $\frac{xdy - ydx}{x^2 + y^2} = dx \Rightarrow d\left(\tan^{-1}\frac{y}{x}\right) = dx$

3. I.F. $= \frac{1}{y^2}$ and $\frac{ye^x dx - e^x dy}{y^2} = d\left(\frac{e^x}{y}\right)$

4. I.F. $= \frac{1}{y}$ and $\frac{x}{y}dy + \log y dx = d\,(x \log y)$

6. $\frac{xdy + ydx}{x^2y^2} = dx \Rightarrow \frac{d(xy)}{(xy)^2} = dx$

7. $\frac{xdy + ydx}{xy} = \frac{dx + dy}{(x+y)^2} \Rightarrow \frac{d(xy)}{xy} = \frac{d(x+y)}{(x+y)^2}$

8. $\frac{xdy - ydx}{4x^2 + y^2} = dy \Rightarrow \frac{(xdy - ydx)/x^2}{4 + (y/x)^2} = dy \Rightarrow \frac{d\left(\frac{y}{x}\right)}{4 + \left(\frac{y}{x}\right)^2} = dy.$

(b) I.F. for a Homogeneous Equation. If $Mdx + Ndy = 0$ is a homogeneous equation in x and y, then $\frac{1}{Mx + Ny}$ is an I.F. provided $Mx + Ny \neq 0$.

Note. If $Mx + Ny$ consists of only one term, use the above method of I.F. otherwise, proceed by putting $y = vx$.

Example. *Solve:* $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$

Sol. The given equation is homogeneous in x and y with

$$M = x^2y - 2xy^2 \quad \text{and} \quad N = -x^3 + 3x^2y$$

Now, $$Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$$

$\therefore$ $$\text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}.$$

Multiplying throughout by $\frac{1}{x^2y^2}$, the given equation becomes $\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$, which is exact. The solution is

$$\int_{y\text{ constant}} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c$$

or $$\frac{x}{y} - 2\log x + 3\log y = c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0$
2. $x^2y\,dx - (x^3 + y^3)\,dy = 0$
3. $(3xy^2 - y^3)\,dx - (2x^2y - xy^2)\,dy = 0$
4. $(x^2 - 3xy + 2y^2)\,dx + x(3x - 2y)dy = 0.$

Answers

1. $\frac{x}{y} - 2\log x + 3\log y = c$
2. $\log y - \frac{x^3}{3y^3} = c$
3. $3\log x - 2\log y + \frac{y}{x} = c$
4. $x^2\log x + 3xy - y^2 = cx^2$

(c) I.F. for an equation of the form $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$.

If $Mdx + Ndy = 0$ is of the form $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$, then $\frac{1}{Mx - Ny}$ is an I.F. provided

$$Mx - Ny \neq 0.$$

Example. *Solve:* $y\,(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0.$

Sol. The given equation is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0.$

Here, $\quad M = xy^2 + 2x^2y^3 \quad$ and $\quad N = x^2y - x^3y^2$

Now, $\quad Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3 \neq 0$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying throughout by $\frac{1}{3x^3y^3}$, the given equation becomes

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

which is exact. The solution is $\displaystyle\int_{y\text{ constant}} \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = c$

or
$$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

or
$$-\frac{1}{xy} + 2\log x - \log y = C, \quad \text{where } C = 3c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $(1 + xy)\,ydx + (1 - xy)xdy = 0.$
2. $(x^2y^2 + xy + 1)\,ydx + (x^2y^2 - xy + 1)xdy = 0.$
3. $y(2xy + 1)dx + x(1 + 2xy - x^3y^3)dy = 0.$
4. $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)\,dy = 0.$

5. $(y - xy^2)dx - (x + x^2y)\, dy = 0.$

6. $(xy \sin xy + \cos xy)\, ydx + (xy \sin xy - \cos xy)\, xdy = 0.$

Answers

1. $-\dfrac{1}{xy} + \log\left(\dfrac{x}{y}\right) = c$

2. $xy + \log\left(\dfrac{x}{y}\right) - \dfrac{1}{xy} = c$

3. $\dfrac{1}{x^2y^2} + \dfrac{1}{3x^3y^3} + \log y = c$

4. $-\dfrac{1}{xy} + 2 \log x - \log y = c$

5. $\log\left(\dfrac{x}{y}\right) - xy = c$

6. $y \cos xy = cx.$

(d) For the equation $Mdx + Ndy = 0$

(i) If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$, a function of x only, then $e^{\int f(x)dx}$ is an I.F.

(ii) If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$, a function of y only, then $e^{\int g(y)dy}$ is an I.F.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $(xy^2 - e^{\frac{1}{x^3}})\, dx - x^2y\, dy = 0.$

Sol. Here, $\quad M = xy^2 - e^{\frac{1}{x^3}} \quad$ and $\quad N = -x^2y$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x},$$ which is a function of x only.

$\therefore$

$$\text{I.F.} = e^{\int -\frac{4}{x} dx} = e^{-4 \log x} = \frac{1}{x^4}$$

Multiplying throughout by $\dfrac{1}{x^4}$, we have $\left(\dfrac{y^2}{x^3} - \dfrac{1}{x^4} e^{\frac{1}{x^3}}\right) dx - \dfrac{y}{x^2}\, dy = 0$

which is exact. The solution is $\displaystyle\int_{y \text{ constant}} \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{\frac{1}{x^3}}\right) dx = c$

or $\quad -\dfrac{y^2}{2x^2} + \dfrac{1}{3}\displaystyle\int -\frac{3}{x^4} e^{\frac{1}{x^3}}\, dx = c \quad$ or $\quad -\dfrac{y^2}{2x^2} + \dfrac{1}{3}\displaystyle\int e^t\, dt = c,\quad$ where $t = \dfrac{1}{x^3}$

or $\quad -\dfrac{y^2}{2x^2} + \dfrac{1}{3} e^t = c \quad$ or $\quad -\dfrac{3y^2}{x^2} + 2e^{\frac{1}{x^3}} = C,\quad$ where $C = 6c.$

Example 2. *Solve:* $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0.$

Sol. Here, $M = xy^3 + y$ and $N = 2x^2y^2 + 2x + 2y^4$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4xy^2 + 2 - 3xy^2 - 1}{xy^3 + y} = \frac{xy^2 + 1}{y(xy^2 + 1)} = \frac{1}{y}$$

which is a function of y only.

$$\therefore \qquad \text{I.F.} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Multiplying throughout by y, we have $(xy^4 + y^2)\,dx + 2\,(x^2y^3 + xy + y^5)\,dy = 0$

which is exact. The solution is $\displaystyle\int_{y \text{ constant}} (xy^4 + y^2)\,dx + \int 2y^5\,dy = c$

or $$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $(x^2 + y^2 + x)\,dx + xydy = 0$
2. $(x^2 + y^2 + 1)\,dx - 2xydy = 0.$
3. $(x^2 + y^2 + 2x)dx + 2ydy = 0.$
4. $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)\,dy = 0.$
5. $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{1}{4}(x + xy^2)dy = 0.$
6. $(x \sec^2 y - x^2 \cos y)dy = (\tan y - 3x^4)dx.$
7. $(xye^{x/y} + y^2)\,dx - x^2e^{x/y}\,dy = 0.$
8. $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0.$
9. $(x^4e^x - 2mxy^2)dx + 2mx^2ydy = 0.$
10. $y(2x^2y + e^x)dx = (e^x + y^3)dy.$
11. $ydx - xdy + \log x\,dx = 0.$
12. $(2x \log x - xy)\,dy + 2ydx = 0.$
13. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)\,dy = 0.$
14. $y \log y\,dx + (x - \log y)\,dy = 0.$
15. $(xy^3 + y)dx + 2\,(x^2y^2 + x + y^4)dy = 0.$

Answers

1. $3x^4 + 6x^2y^2 + 4x^3 = c$
2. $x - \frac{y^2}{x} - \frac{1}{x} = c$
3. $e^x(x^2 + y^2) = c$
4. $\left(y + \frac{2}{y^2}\right)x + y^2 = c$
5. $3x^2y + x^4y^3 + x^6 = c$
6. $\frac{\tan y}{x} + x^3 - \sin y = c$
7. $e^{x/y} + \log x = c$
8. $x^2y(x - ay) = c$
9. $e^x + m\left(\frac{y}{x}\right)^2 = c$
10. $\frac{2x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = c$
11. $1 + y + \log x = cx$
12. $2y \log x - \frac{1}{2}y^2 = c$
13. $x^3y^2 + \frac{x^2}{y} = c$
14. $x \log y - \frac{1}{2}(\log y)^2 = c$
15. $\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c.$

III Formula for I.F

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$, is a function of x only, then

I.F $e^{\int f(x)dx}$

IV Formula for I.F

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$$, is a function of y only, then

I.F $e^{-\int g(y)\,dy}$

Questions

1. $(x^2+y^2+2x)\,dx + 2y\,dy = 0$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Find Integrating factor

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-0}{2y} = 1$$

$\Rightarrow$ I.F $e^{\int 1\,dx} = e^x$

multibly given equation by e^x to obtain it exact

$e^x(x^2+y^2+2x)\,dx + e^x\,2y\,dy = 0$

$M = e^x x^2 + e^x y^2 + e^x\,2x$

$N = e^x\,2y$

$\frac{\partial M}{\partial y} = 0 + 2ye^x \qquad \frac{\partial N}{\partial x} = 2ye^x$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ equation is exact

$\int (e^x x^2 + e^x y^2 + e^x 2x)\,dx + \int 0\,dy = 0$

$y =$ constant

[Using Formula $\int e^x f(x)dx + e^x f'(x)\,dx = e^x f(x)$]

$$e^x y^2 + e^x x^2 = c$$

$$\Rightarrow e^x(x^2 + y^2) = c$$

2. $(x\sec^2 y - x^2\cos y)\,dy = (\tan y - 3x^4)\,dx$

$(\tan y - 3x^4)\,dx - (x\sec^2 y - x^2\cos y)\,dy = 0$

$\frac{\partial M}{\partial y} = \sec^2 y \qquad \frac{\partial N}{\partial x} = -\sec^2 y + 2x\cos y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

I.F $\quad \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{\sec^2 y + \sec^2 y - 2x\cos y}{-x(\sec^2 y \mp x\cos y)}$

$$= \frac{2(\cancel{\sec^2 y - x\cos y})}{-x(\cancel{\sec^2 y - x\cos y})}$$

I.F $e^{-\int \frac{2}{x}dx} = e^{-2\log x} = e^{-\log x^2} = x^{-2} = \frac{1}{x^2}$

multiply given equation by x^{-2} or $\frac{1}{x^2}$

multiply given equation by $\frac{1}{x^2}$

$$\frac{1}{x^2}(\tan y - 3x^4)\,dx - \frac{1}{x^2}(x\sec^2 y - x^2\cos y)\,dy = 0$$

$$M = \frac{\tan y}{x^2} - 3x^2 \qquad N = -\frac{\sec^2 y}{x} + \cos y$$

$$\frac{\partial M}{\partial y} = \frac{\sec^2 y}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{x^2}\sec^2 y$$

now equation became exact, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution is

$$\int_{y=\text{constant}} \left(\frac{\tan y}{x^2} - 3x^2\right) dx + \int \cos y\, dy = c$$

$$-\frac{\tan y}{x} - \frac{3x^3}{3} + \sin y = c$$

3. $(xye^{x/y} + y^2)\,dx - x^2 e^{x/y}\,dy = 0$

$$M = xye^{x/y} + y^2 \qquad N = -x^2 e^{x/y}$$

$$\frac{\partial M}{\partial y} = x\left[1\cdot e^{x/y} + ye^{x/y}\left(-\frac{x}{y^2}\right)\right] + 2y$$

$$= xe^{x/y} - \frac{x^2}{y}e^{x/y} + 2y \qquad -(1)$$

$$\frac{\partial N}{\partial x} = -\left[2xe^{x/y} + x^2 e^{x/y}\frac{1}{y}\right]$$

$$= -\left[2x\, e^{x/y} + \frac{x^2}{y}e^{x/y}\right] \qquad -(2)$$

From (1) and (2)

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Find Integrating factor

$\partial y \quad \partial x$

3. $(x^4 e^x - 2mxy^2)\,dx + 2mx^2y\,dy = 0$

$M = x^4e^x - 2mxy^2 \qquad N = 2mx^2y$

$\frac{\partial M}{\partial y} = 0 - 4mxy \qquad \frac{\partial N}{\partial x} = 4mxy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Integrating factor

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -4mxy - 4mxy = -8mxy$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-\cancel{8}^{4}mxy}{\cancel{2}mx^2y} = -\frac{4}{x}$

$\Rightarrow$ I.F $e^{\int -\frac{4}{x}dx}$

$= e^{-4\log x}$

$= e^{+\log x^{-4}}$

$= \frac{1}{x^4}$

multibly given equation by $\frac{1}{x^4}$

$\frac{1}{x^4}(x^4e^x - 2mxy^2)\,dx + \frac{1}{x^4}2mx^2y\,dy = 0$

$(e^x - 2my^2/x^3)\,dx + (2my/x^2)\,dy = 0$

$M = e^x - \frac{2my^2}{x^3}$ $\quad N = \frac{2my}{x^2}$

$\frac{\partial M}{\partial y} = 0 - \frac{4my}{x^3}$ $\quad \frac{\partial N}{\partial x} = -\frac{4my}{x^3}$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, now equation become exact

its Solution is $\int\limits_{y = \text{constent}} \left(e^x - \frac{2my^2}{x^3}\right) dx + \int (\text{terms in } N \text{ not Contain } x)\, dy = c$

$= e^x - 2my^2 \frac{x^{-3+1}}{-3+1} + 0 = c$

$= e^x + \frac{\cancel{2}my^2}{\cancel{2}} \frac{1}{x^2} = c$

$e^x + m\left(\frac{y}{x}\right)^2 = c$

4. $y \log y \, dx + (x - \log y) \cdot dy = 0$

$M = y \log y$ $\quad N = x - \log y$

$\frac{\partial M}{\partial y} = 1 \cdot \log y + \frac{y}{y} = 1 + \log y$

$\frac{\partial N}{\partial x} = 1 - 0 = 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

I.f $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{} = \dfrac{\cancel{1} + \log y - \cancel{1}}{y \log y} = \frac{1}{y}$

I.f $e^{-\int \frac{1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = 1/y$

$\frac{1}{y}\,[y\log y]\,dx + \frac{x}{y} - \log y/y\ dy = 0$

$M = \log y \qquad N = \frac{x}{y} - \frac{\log}{y}$

$\frac{\partial M}{\partial y} = \frac{1}{y} \qquad \frac{\partial N}{\partial x} = \frac{1}{y}$

$\Rightarrow \frac{\partial M}{\partial y} \equiv \frac{\partial N}{\partial x}$, enact equation and

its solution is

$\int\limits_{y=\text{Canstent}} \log y\ dx + \int -\frac{\log y}{y}\ dy = 0$

$x \cdot \log y - \frac{(\log y)^2}{2} = c$

5. $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$

$\frac{\partial M}{\partial y} = 3xy^2 + 1 \qquad \frac{\partial N}{\partial x} = 4xy^2 + 2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Find Integrating Factor

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3xy^2 + 1 - 4xy^2 - 2}{y(xy^2+1)}$

$= \frac{-xy^2 - 1}{y(xy^2+1)} = -\frac{1}{y}$

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I.F $e^{-\int -\frac{1}{y} dy} = e^{\int \frac{1}{y} dy}$

$= e^{\log y} = y$

multibly given equation by y

$(xy^4 + y^2) dx + 2(x^2y^3 + xy + y^5) dy = 0$

$\frac{\partial M}{\partial y} = 4xy^3 + 2y \qquad \frac{\partial N}{\partial x} = 4xy^3 + 2y$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, thus equation become enact

now find its Solution

$\int_{y=\text{Constent}} (xy^4 + y^2) dx + \int 2y^5 dy = 0$

$\frac{x^2y^4}{2} + xy^2 + \frac{2y^6}{6} = c$

$\frac{x^2y^4}{2} + xy^2 + \frac{y^3}{3} = c$

Exact Equation

Q $(x+y)^2\left(x\frac{dy}{dx}+y\right)=xy\left(1+\frac{dy}{dx}\right)$

Solution

$$(x+y)^2\left(\frac{x\,dy+y\,dx}{dx}\right)=xy\left(\frac{dx+dy}{dx}\right)$$

$$(x+y)^2(x\,dy+y\,dx)=xy(dx+dy)$$

$$\frac{x\,dy+y\,dx}{xy}=\frac{dx+dy}{(x+y)^2}$$

$$d[\log(xy)]=\frac{dx+dy}{(x+y)^2}$$

Taking integration on both sides

$$\log(xy)=-\frac{1}{(x+y)}+C$$